The poisson processes

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1 Introduction

The Poisson process is one of the most widely-used counting processes. It is usually used in scenarios where we are counting the occurrences of certain events that appear to happen at a certain rate, but completely at random (without a certain structure). For example, suppose that from historical data, we know that earthquakes occur in a certain area with a rate of 2 per month. Other than this information, the timings of earthquakes seem to be completely random. Thus, we conclude that the Poisson process might be a good model for earthquakes. In practice, the Poisson process or its extensions have been used to model

- the number of car accidents at a site or in an area;
- the location of users in a wireless network;
- the requests for individual documents on a web server;
- the outbreak of wars;
- photons landing on a photodiode.

1.1 Properties

Here, we briefly review some properties of the Poisson random variable that we have discussed in the previous chapters. Remember that a discrete random variable X is said to be a Poisson random variable with parameter μ, shown as $X \sim \text{Poisson}(\mu)$, if its range is $R_X = \{0, 1, 2, 3, ...\}$, and its PMF is given by

$$P_X(k) = \begin{cases} e^{-\mu}\frac{\mu^k}{k!} & \text{for } k \in R_X, \\ 0 & \text{otherwise} \end{cases}$$

Here are some useful facts that we have seen before: If $X \sim \text{Poisson}(\mu)$, then $\mathrm{E}[X] = \mu$, and $\mathrm{Var}(X) = \mu$. If $X_i \sim \text{Poisson}(\mu_i)$, for $i = 1, 2, \ldots, n$, and the X_i's are independent, then

$$X_1 + X_2 + \ldots + X_n \sim \text{Poisson}(\mu_1 + \mu_2 + \ldots + \mu_n).$$

The Poisson distribution can be viewed as the limit of the binomial distribution.

Theorem

Let $Y_n \sim \text{Binomial}(n, p = p(n))$. Let $\mu > 0$ be a fixed real number, and $\lim_{n\to\infty} np = \mu$. Then, the PMF of Y_n converges to a Poisson(μ) PMF as $n \to \infty$. That is, for any $k \in \{0, 1, 2, ...\}$, we have

$$\lim_{n\to\infty} P_{Y_n}(k) = \frac{e^{-\mu}\mu^k}{k!}.$$

Definition of the Poisson Process:

The above construction can be made mathematically rigorous. The resulting random process is called a Poisson process with rate (or intensity) λ. Here is a formal definition of the Poisson process.

The Poisson Process

Let $\lambda > 0$ be fixed. The counting process $\{N(t), t \in [0, \infty)\}$ is called a Poisson process with rate λ if all the following conditions hold:

1. $N(0) = 0$;
2. $N(t)$ has independent increments;
3. The number of arrivals in any interval of length $\tau > 0$ has Poisson($\lambda\tau$) distribution.

Note that from the above definition, we conclude that in a Poisson process, the distribution of the number of arrivals in any interval depends only on the length of the interval, and not on the exact location of the interval on the real line. Therefore, the Poisson process has stationary increments.

1.2 Simulations

In the following simulation, we consider M servers subject to attacks during a period of time T (for instance, 1 year). Divide the interval T into N subintervals of size T/N, and in each of these, suppose that an attack can occur with a probability of $\lambda T/N$. The simulation is an approximation of a Poisson process.

We have:

- **Independence of attacks**: In the Poisson process, events (attacks in your case) must be independent, meaning that the probability of an attack in a certain time interval does not influence the probability of an attack in another interval.
- **Constancy of probability**: In the Poisson process, the probability of a certain number of events in a fixed time interval must be constant. The formula used in the simulation approximates this constancy by using the probability $\lambda T/N$.

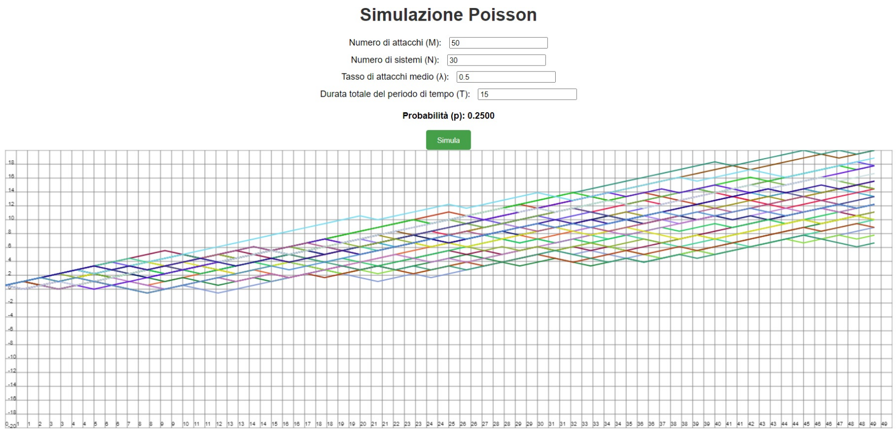

Figure 1: Poisson Process

- **Finite number of systems and attacks**: We are considering a finite number of systems and a finite number of total attacks. Therefore, while the Poisson distribution represents an infinite number of events in an infinite interval, the simulation represents a finite number of attacks in a finite number of systems during a finite period of time.

The representation is an approximation as we are not directly dealing with a Poisson distribution. However, in practical contexts, this approximation can be sufficient for modeling stochastic phenomena, especially when the number of events is large and the probability of a single event is small.

Try the simulation here (http://wendy.altervista.org/poisson.html)

source: https://shorturl.at/wLQR3