The Wiener process and the GBM

Wendy Rosettini

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1 Introduction

One of the most important stochastic processes is the Wiener process[1] or Brownian (motion) process. he Wiener process can be considered a continuous version of the simple random walk. This continuous-time stochastic process is a highly studied and used object. It plays a key role different probability fields, particularly those focused on stochastic processes such as stochastic calculus and the theories of Markov processes, martingales, Gaussian processes, and Levy processes.

The Wiener process is named after Norbert Wiener, but it is called the Brownian motion process or often just Brownian motion due to its historical connection as a model for Brownian movement in liquids, a physical phenomenon observed by Robert Brown. But the physical process is not true a Wiener process, which can be treated as an idealized model. The Wiener process is

arguably the most important stochastic process. The other important stochastic process is the Poisson (stochastic) process.

2 Definition

A real-valued stochastic process $\{W_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ is a standard Wiener (or Brownian motion) process if it has the following properties:

1. The initial value of the stochastic process $\{W_t : t \geq 0\}$ is zero with probability one, meaning $P(W_0 = 0) = 1$.
2. The increment $W_t - W_s$ is independent of the past, that is, W_u, where $0 \leq u \leq s$.
3. The increment $W_t - W_s$ is a normal variable with mean 0 and variance $t - s$.

[1] https://hpaulkeeler.com/wiener-or-brownian-motion-process/

In some literature, the initial value of the stochastic process may not be given. Alternatively, it is simply stated as $W_0 = 0$ instead of the more precise (probabilistic) statement given above.

Also, some definitions of this stochastic process include an extra property or two. For example, from the above definition, we can infer that increments of the standard Wiener process are stationary due to the properties of the normal distribution. But a definition may include something like the following property, which explicitly states that this stochastic process is stationary:

- For $0 \leq u \leq s$, the increment $W_t - W_s$ is equal in distribution to W_{t-s}.

The definitions may also describe the continuity of the realizations of the stochastic process, known as sample paths, which we will cover in the next section.

It's interesting to compare these defining properties with the corresponding ones of the homogeneous Poisson stochastic process. Both stochastic processes build upon divisible probability distributions. Using this property, Lévy processes generalize these two stochastic processes.

3 Derivations of Wiener Process

- 1. Initial Value:

  The initial value being 0 simply reflects the starting point of the process. This is often considered a natural choice, and it aligns with the idea that the process represents a particle starting at the origin.

- 2. Independent Increments:

  The independence of increments is a key property. It means that the future behavior of the process does not depend on its past. This property is often derived using the fact that normal distributions are stable under linear combinations.

- 3. Normal Increments:

  The normal distribution of increments is a consequence of the Central Limit Theorem. As you sum a large number of independent, identically distributed random variables (in this case, increments), the distribution tends to approach a normal distribution. This is a fundamental result in probability theory.

4 Simulations

The JavaScript code[2] simulates the Wiener process. The simulation involves generating normally distributed random increments. At each time

[2] http://wendy.altervista.org/wienerprocess.html

step, an increment is randomly drawn and added to the current value, thus simulating the continuous and random change over time in the Wiener process. This process is known for its property of having independently and normally distributed increments.

The result of the simulation is then displayed in an interactive chart. The chart depicts the simulated path of the Wiener process over time, providing a visual illustration of the erratic and variable nature of Brownian motion.

Each execution of the simulation produces a unique path, highlighting the stochastic nature of the process. The interactivity of the chart allows users to explore the simulated path, offering a dynamic experience in understanding the behavior of the Wiener process.

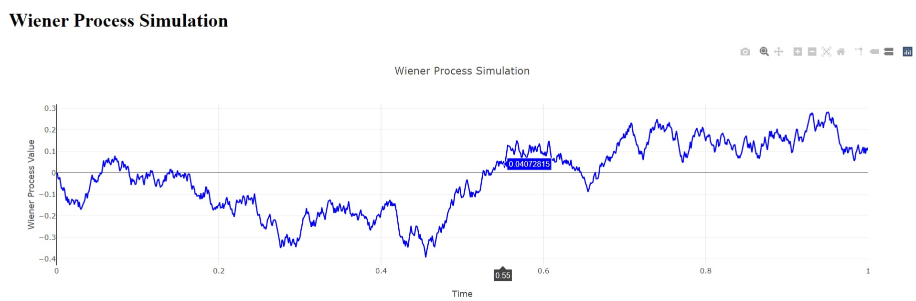

Figure 1: Wiener process