The Ito integral and calculus

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1 Introduction

Itô calculus[1], named after Kiyosi Itô, extends traditional calculus to handle stochastic processes, specifically Brownian motion. This calculus is crucial in mathematical finance and stochastic differential equations. The key concept is the Itô stochastic integral, a stochastic version of the Riemann–Stieltjes integral. In this integral, both the integrands and integrators are stochastic processes.

The integral is represented as:

$$Y_t = \int_0^t H_s \, dX_s,$$

where H is a locally square-integrable process adapted to the filtration generated by X, a Brownian motion, or more generally, a semimartingale. The result is another stochastic process. Unlike traditional calculus, Brownian motion paths lack the properties necessary for standard calculus techniques. With the integrand being a stochastic process, the Itô stochastic integral involves integrating with respect to a non-differentiable function with infinite variation over time intervals.

The insight lies in defining the integral as long as the integrand H is adapted, meaning its value at time t depends only on information available up to that time. The integral is approached by choosing a sequence of partitions, constructing Riemann sums, and taking the limit in probability as the partition mesh approaches zero.

Key results of Itô calculus include the integration by parts formula and Itô's lemma, a change of variables formula with quadratic variation terms. In mathematical finance, the integral is conceptualized as a continuous-time trading strategy. The integrand represents the amount of stock held, the integrator models price movements, and the integral reflects the total value, including the stock's worth. Stock prices are often modeled by stochastic processes like Brownian motion or geometric Brownian motion. The adapted condition ensures the trading strategy uses only available information, preventing unrealistic gains through clairvoyance. Additionally, it ensures the stochastic integral does not diverge when calculated using limit techniques.

[1] https://shorturl.at/I0348

2 Simulations

The simulation[2] implements a simplified example of Itô integration using Brownian motion. Initially, random Brownian motion increments are generated through the 'generateBrownianIncrements' function. Subsequently, the 'simulateItoIntegral' function approximates the Itô integral by multiplying the increments with the values of a specified integrand function. In this example, the integrand function is t^2. Finally, the approximate result is displayed in the HTML page. The variability of the result depends on the random nature of the generated Brownian increments. By modifying the integrand function or the number of increments, different scenarios of stochastic integration can be explored.

Result of Itô integral simulation: -225490.7206

2.1 Code

```
<script>
  // Function to generate Brownian motion increments
  function generateBrownianIncrements(numIncrements) {
    const dt = 1 / numIncrements;
    const increments = [];

    for (let i = 0; i < numIncrements; i++) {
      const dW = Math.sqrt(dt) * (2 * Math.random() - 1);
      increments.push(dW);
    }

    return increments;
  }

  // Function to simulate the Itô integral
  function simulateItoIntegral(integrand, brownianIncrements) {
    let integral = 0;

    for (let i = 0; i < brownianIncrements.length; i++) {
      integral += integrand(i) * brownianIncrements[i];
    }

    return integral;
  }

  // Example integrand function (you can replace this with your own)
  function integrandFunction(t) {
    return t * t;
  }

  // Number of increments in the simulation
  const numIncrements = 1000;

  // Generate Brownian motion increments
  const brownianIncrements = generateBrownianIncrements(numIncrements);

  // Simulate Itô integral
  const result = simulateItoIntegral(integrandFunction, brownianIncrements);

  // Display the result
  document.write(`<p>Result of Itô integral simulation: ${result.toFixed(4)}</p>`);
</script>
```

[2] http://wendy.altervista.org/ito.html