SDE

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1 Introduction

Stochastic Processes

A stochastic process is a mathematical model that describes the evolution of a system over time in a probabilistic manner. Unlike deterministic processes, where the future behavior is completely determined by the initial conditions, stochastic processes involve randomness. A stochastic process is typically defined by a collection of random variables, indexed by time or another parameter.

Types of Stochastic Processes

1. **Discrete-Time Stochastic Process:**
    - Examples include random walks and Markov chains.
    - The process evolves in discrete time steps.
2. **Continuous-Time Stochastic Process:**
    - Examples include Brownian motion and Poisson processes.
    - The process evolves continuously over time.

Stochastic Differential Equations (SDEs)

Stochastic Differential Equations are differential equations that involve both deterministic and random components. They are used to model systems where randomness plays a significant role in the evolution of the system. The general form of an SDE is:

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t$$

where:

- X_t is the stochastic process.
- $a(t, X_t)$ and $b(t, X_t)$ are functions that determine the drift and diffusion of the process.

- dW_t is the differential of a Wiener process (Brownian motion), representing the random component.

Components of SDEs

1. **Drift Term ($a(t, X_t)$):**
   - Represents the deterministic part of the change in the process.
2. **Diffusion Term ($b(t, X_t)$):**
   - Represents the random part of the change, scaled by the Wiener process.
3. **Solution of SDE:**
   - The solution is often given in terms of a stochastic integral, which involves integrating with respect to the Wiener process.

Itô's Lemma

Itô's Lemma is a key result used when dealing with SDEs. It is an extension of the chain rule for functions of stochastic processes.

$$dY_t = f(t, X_t)\,dt + g(t, X_t)\,dW_t$$

$$\Rightarrow d(f(t, X_t)) = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial x^2}\right)dt + b\frac{\partial f}{\partial x}\,dW_t$$

Applications

1. **Finance:**
   - SDEs are widely used to model the dynamics of financial instruments, such as stock prices.
2. **Physics:**
   - Brownian motion is used to model the random movement of particles in a fluid.
3. **Biology:**
   - Used to model population growth, disease spread, and other biological phenomena.
4. **Engineering:**
   - SDEs are applied to study the reliability and performance of systems subjected to random factors.

Stochastic processes and SDEs provide a powerful framework for understanding and modeling systems in the presence of uncertainty and randomness. They are essential tools in both theoretical and applied mathematics.

2 Simulations

I realized a simulator`http://wendy.altervista.org/SDE%20SIMULATOR/SDE_simulator.html` of SDE's. Here, you can see the web-only version that allows you to explore any useful stochastic process. In the upper part of the website, there is the general scheme where M systems are subject to a series of N attacks. On the x-axis, we indicate the attacks and on the Y-axis, we simulate the accumulation of a "security score" (-1, 1), where the score is -1 if the system is penetrated and 1 if the system was successfully "shielded" or protected.

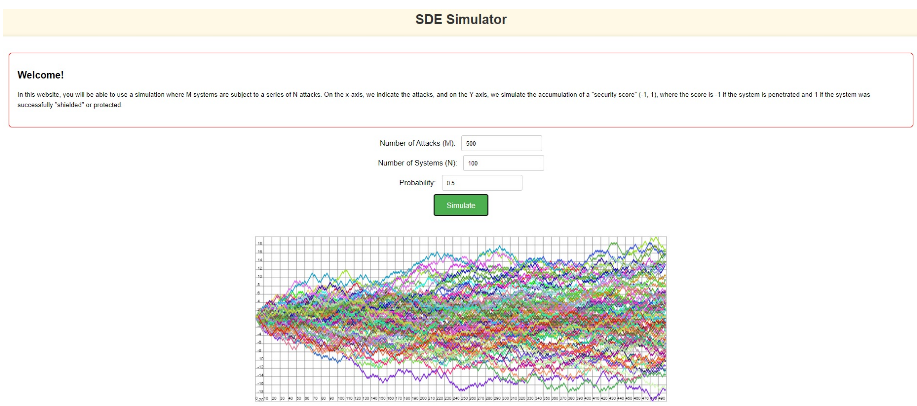

Figure 1: SDE simulator

Then the website allows you to select the Stochastic Differential Equation, that will modify the above graph, according to the differential equation chosen.

Arithmetic Brownian Motion is a mathematical model used to describe the random motion of a particle, typically in the context of financial mathematics and the modeling of asset prices. It is a specific type of stochastic process where the change in the value of the variable over a small time interval follows a normal distribution.

The `makeBrownianMotion` function generates simulated data for arithmetic Brownian motion. It takes as input the number of time points (`attacchi`), the number of systems (`sistemi`), the average drift rate (`mu`), and the volatility (`sigma`). It returns a matrix of simulated data, where each row represents a system, and each column represents a time point.

The data generation follows the formula of arithmetic Brownian motion. For each system and time point, it calculates an increment based on the drift rate and volatility. Subsequently, it updates the matrix with the new values by adding the increment to the previous value.

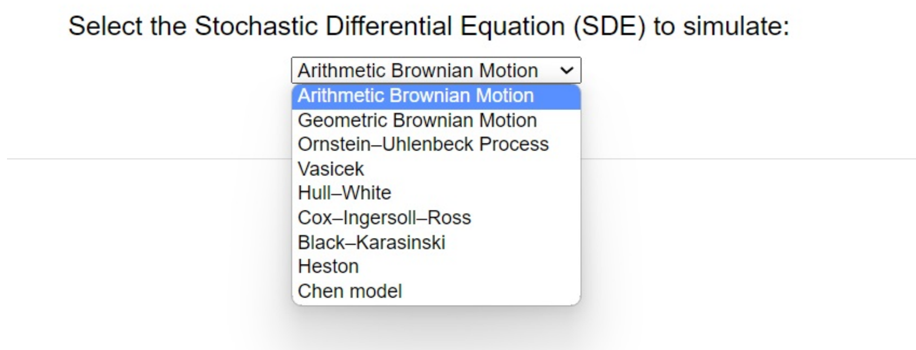

Figure 2: Selected Stochastic Differential Equation

For example, let's see the Arithmetic Brownian Motion. Arithmetic Brownian Motion is a mathematical model used to describe the random motion of a particle, typically in the context of financial mathematics and the modeling of asset prices. It is a specific type of stochastic process where the change in the value of the variable over a small time interval follows a normal distribution.

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Observing the Brownian Motion graph generated by the simulation, we see trajectories moving in a disorderly manner, with fluctuations resulting from the random variations introduced by the stochastic term.

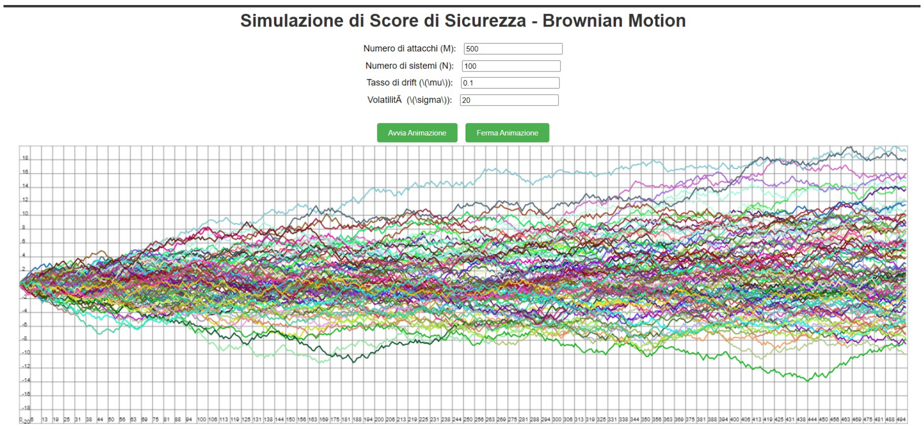

Figure 3: Brownian Motion